Chaper 5 Logarithmic,Exponential and Othter Transcendental Functions

5. 1. The Natural Logarithmic Function and Differentiation

Difinition of Natural Logarithmic Function
The Natural Logarithmic Function is defined by

$$\ln x = \int_1^x 1/t \cdot dt \quad , x > 0$$

Logarithmic properties.
If a & b are positive numbers and n is rational ,then following properties are true.

1. $\ln(1) = 0$
2. $\ln(a \cdot b) = \ln a + \ln b$
3. $\ln(a^n) = n \cdot \ln a$
4. $\ln(a / b) = \ln a - \ln b$.

Ex: expanding the function of $f(x) = \ln \dfrac{(x^2+3)^2}{x \cdot \sqrt[3]{x^2+1}}$

Sol:原式$= \ln(x^2+3)^2 - \ln(x \cdot \sqrt[3]{x^2+1})$

$$= 2 \cdot \ln(x^2+3) - \left[\ln x + \ln(x^2+1)^{\frac{1}{3}} \right]$$

$$= 2 \cdot \ln(x^2+3) - \text{in}x - \frac{1}{3}\ln(x^2+1)$$

Derivative of the Natural Logarithmic Function
Let u be differentiable function of x.

1. $d/dx\,[\ln x] = 1/x \,, x > 0$
2. $d/dx\,[\ln u] = \dfrac{1}{u} \cdot \dfrac{du}{dx} = u'/u \,, u > 0$

Ex.:Differentiation $\frac{d}{dx}\left[\ln(x^2+1)\right]$

Sol: $3\left[\ln(x^2+1)^2\right]^2 \cdot \frac{1}{x^2+1} \cdot 2x = \frac{6x}{x^2+1}\left[\ln(x^2+1)\right]^2$

5.2 The Natural Logarithmic Function and Integration

Let u be a differentiable function of x.

1. $\int \frac{1}{x} \cdot dx = \ln|x| + c$
2. $\int \frac{1}{u} \cdot du = \ln|u| + c$

Ex1: $\int \frac{2}{x} \cdot dx = 2\int \frac{1}{x} \cdot dx = 2\ln x + c$

Ex2: $\int \cot x \cdot dx$

$= \int \frac{\cos x}{\sin x} \cdot dx = \int \frac{u}{du} = \ln|u| + c \qquad du = \cos x, u = \sin x$

Integrals of six Basic Trigonometric Functions.

1. $\int \sin u \cdot du = -\cos u + c$
2. $\int \cos u \cdot du = \sin u + c$
3. $\int \tan u \cdot du = \ln|\cos u| + c$
4. $\int \cot u \cdot du = \ln|\sin u| + c$
5. $\int \sec u \cdot du = \ln|\sec u + \tan u| + c$
6. $\int \csc u \cdot du = -\ln|\csc u + \cot u| + c$

Ex: $\int_0^{\frac{4}{\pi}} \sqrt{1+\tan^2 x} \cdot dx$

Sol:原式 $= \int_0^{\frac{4}{\pi}} \sec x \cdot dx = \left[\ln\left|\sec x + \tan x\right|\right]_0^{\frac{4}{\pi}}$

$$= \left[\ln\left|\sec\frac{4}{\pi} + \tan\frac{4}{\pi}\right|\right] - \left[\ln\left|\sec 0 + \tan 0\right|\right]$$

$$= \ln\left|\sqrt{2}+1\right| + \ln|1| = \ln\left(\sqrt{2}+1\right)$$

5.3 Inverse Functions

Definition of Inverse Functions $f^{-1}(x) = g(x) \quad f(g(x) = x$

A function g is the inverse of the function f if
$f(g(x)) = x$ **for each x in the domain of g**
and
$g(f(x)) = x$ **for each x in the domain of f**

Ex: show that the following functions are inverses of each other

$f(x) = 2x^3 - 1$ **and** $g(x) = \sqrt[3]{\frac{x+1}{2}}$

Sol:

$$f(g(x)) = f(\sqrt[3]{\frac{x+1}{2}}) = 2 \cdot (\sqrt[3]{\frac{x+1}{2}})^3 - 1 = x$$

$$g(f(x)) = g(2x^2 - 1) = \sqrt[3]{\frac{(2x^3+1)+1}{2}} = x$$

5.4 Exponetial Function :Differentiation and Integration

Definition of the natural exponential functions The inverse ofnatural unction f(x) = ln x is Called the natural exponential function and denoted by $f^{-1}(x) = e^x$

Ex:Find the x from $e^{x+1} = 7$

Sol:

$$\ln(e^{x+1}) = \ln 7$$
$$x + 1 = 7$$
$$\therefore x = \ln 7 - 1 = 0.946$$

5.5 Bases othor than e and Appliications

Definition of Logarithmic Function to Base a
If a is a positive real number ($a \neq 1$) and x is any positive real Number ,then Logrithmic Function to the base a is denoted by $\log_a x$ and is defined as :

$$\log_a x = \frac{1}{\ln a} \cdot \ln x$$

Some propertues:

1. $\log_a 1 = 0 \qquad \therefore \frac{1}{\ln a} \cdot \ln 1 = 0$

2. $\log_a x \cdot y = \log_a x + \log_a y$

$$\because \frac{1}{\ln a} \ln(x \cdot y)$$
$$= \frac{1}{\ln a} \cdot \ln x + \frac{1}{\ln a} \ln y$$
$$= \log_a x + \log_a y$$

3. $\log_a (x/y) = \log_a x - \log_a y$

4. $\log_a x^n = n \cdot \log_a x$

Ex: solve for x in the following equations

$$3^x = \frac{1}{27}$$

Sol:

$$3^x = 1/27$$

$$\log_3 \cdot 3^x = \log_3(\frac{1}{27})$$

$$x = \log_3(3^{-3}) = -3$$

5.6 Differential Equations :Growth and Decay

Ex: solve the differential equation. $y' = \frac{2x}{y}$

Slo:

$$dy/dx = 2x/y$$

$$y \cdot dy = 2x \cdot dx$$

$$\int y \cdot dy = 2 \cdot \int x \cdot dx$$

$$\frac{1}{2}y^2 = x^2 + C_1$$

$$y^2 - 2x^2 = C$$

5.7 Differential Equations:Separation of Variables

Sparation of Variables

$$M(x) + N(y) \cdot dy/dx = 0$$

$$M(x) \cdot dx = -N(y) \cdot dy$$

Ex: Find the greral solution of $(x^2 + 4) \cdot dy/dx = x \cdot y$

Sol: $(x^2+4)\cdot {}^{dy}\!/\!_{dx} = x\cdot y$

變數分離→ $\frac{1}{y}\cdot dy = (\frac{x}{x^2+4})\cdot dx$

$$\int \frac{1}{y}\cdot dy = \int (\frac{x}{x^2+4})\cdot dx$$

$$\ln|y| = \int \frac{d(x^2+4)}{(x^2+4)}\cdot\frac{1}{2} = \frac{1}{2}\ln(x^2+4)+C_1$$

$$|y| = \exp\left[\frac{1}{2}\ln(x^2+4)+C_1\right] = \sqrt{x^2+4}\cdot e^{c_1}$$

$$|y| = \pm e^{c_1}\cdot\sqrt{x^2+4}$$

$$y = C\cdot\sqrt{x^2+4}$$

為 General Solution

5.8 Inverse Trigonometric Functions and Differentiation

Ex: Differentiate $y = \sin^{-1}x + x\cdot\sqrt{1-x^2}$

$${}^{dy}\!/\!_{dx} = \frac{1}{\sqrt{1-x^2}} + x\cdot\frac{1}{2}\cdot(1-x^2)^{-\frac{1}{2}}\cdot(-2x)+\sqrt{1-x^2}$$

$$= \frac{1}{\sqrt{1-x^2}} - \frac{x^2}{\sqrt{1-x^2}} \neq \sqrt{1-x^2}$$

$$= 2\cdot\sqrt{1-x^2}$$

5.9 Inverse Trigonometric Functions and Intergration

Let u be a differentiable function of x ，and Let $a>0$

1. $\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}(\frac{u}{a})+C$ 2. $\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}(\frac{u}{a})+C_1$

3. $\int \frac{du}{u\cdot\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}(\frac{|u|}{a})+C_1$

5.10 Hyperbolic Function : Differentiation & Integration.

Defn of Hyperbolic Functions.

$$\sin x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{1}{\tanh x}, x \neq 0$$

$$\sec hx = \frac{1}{\cosh x} \qquad \csc x = \frac{1}{\sinh x}, x \neq 0$$

Caper6 Applications of Integration

Area of Region Between Two Curves

If f & g are Continuous on $[a,b]$ $\& g(x) \leq f(x)$ for all x in $[a,b]$，then the region bounded by the graphs of f&g and the vertical lines
X=a & x=b is

$$A = \int_a^b (f(x) - g(x)) \cdot dx$$

Ex: Find the area of the region bounded by graphs of

$$y = x^2 + 2, y = -x, x = 0 \quad x = 0 \text{ and } x = 1$$

$$\int_0^1 \left[\left(x^2 + 2\right) + x \right] \cdot dx$$

Sol:
$$= \left[\frac{x^3}{3} + \frac{x^2}{2} + 2x \right]_0^1 = \frac{1}{3} + \frac{1}{2} + 2 = \frac{17}{6}$$

Ex: Find the arc length of the graph of

$$y=\frac{x^3}{6}+\frac{1}{2x} \quad \text{on the interval } \left[\frac{1}{2},2\right]$$

Sol:

$$dy/dx = 3x^2/6 + \frac{1}{2}(-1)x^{-2}$$

$$= x^2/2 - \frac{1}{2}x^2 = \frac{1}{2}(x^2 - \frac{1}{x^2})$$

$$S=\int_{\frac{1}{2}}^{2}\sqrt{1+\frac{1}{4}(x^2+\frac{1}{x^2})^2}\cdot dx$$

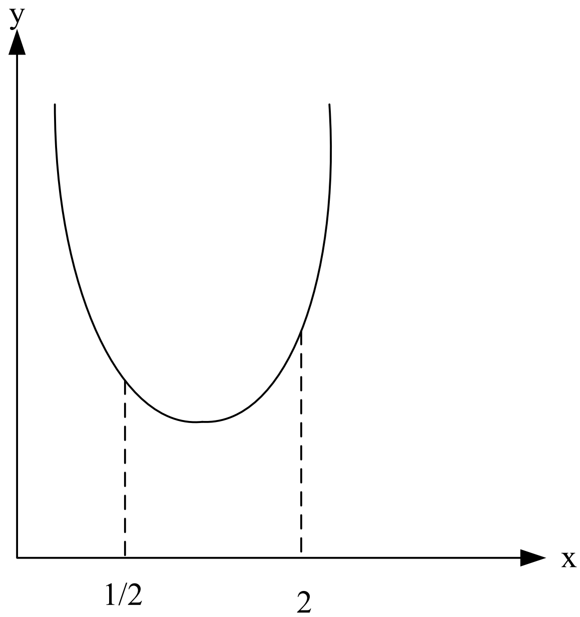
$$=\int_{\frac{1}{2}}^{2}\sqrt{\frac{1}{4}(x^4-2+\frac{1}{x^4})+1}\cdot dx$$

$$=\int_{\frac{1}{2}}^{2}\sqrt{\frac{1}{4}(x^4-2+\frac{1}{x^4})}\cdot dx$$

$$=\int_{\frac{1}{2}}^{2}\frac{1}{2}\sqrt{(x^2+\frac{1}{x^2})}\cdot dx$$

$$=\frac{1}{2}\int_{\frac{1}{2}}^{2}(x^2+\frac{1}{x^2})\cdot dx$$

$$=\frac{1}{2}\left[\frac{x^3}{3}-\frac{1}{x}\right]_{\frac{1}{2}}^{1}=\frac{33}{16}$$



Ex: Find the centroid of the region bounded by the graphs of

$$f(x)=4-x^2 \ \& \ g(x)=x+2$$

Sol:求交點

$$4-x^2=x-2$$

$$x^2+x-2=0$$

$$(x+2)(x-1)=0$$

$$x=-2, x=1$$

$$A=\int_{-2}^{1}[f(x)-g(x)]\cdot dx$$

$$=\int_{-2}^{1}\left[4-x^2-x-2\right]\cdot dx$$

$$=\int_{-2}^{1}\left(-x^2-x+2\right)\cdot dx=\frac{9}{2}$$

$$\bar{x}=\frac{1}{A}\cdot\int_{-2}^{1}x\cdot[f(x)-g(x)]\cdot dx$$

$$=\frac{2}{9}\cdot\int_{-2}^{1}x\left(-x^2-x+2\right)\cdot dx=\frac{2}{9}\cdot\int_{-2}^{1}\left(-x^3-x^2+2x\right)\cdot dx=-\frac{1}{2}$$

Chapter 7 Integration Technigues , Lhopitals Rule , and Improper Integrals

Basic Integration Rule (a > 0)

1. $\int kf(u)\cdot du=k\cdot\int f(u)\cdot du$

2. $\int (f(u)\pm g(u))\cdot du=\int f(u)\cdot du\pm\int g(u)\cdot du$

3. $\int du=u+c$

4. $\int u^n\cdot du=\frac{u^{n+1}}{n+1}+C$

5. $\int\frac{du}{u}=\ln|u|+C$

6. $\int e^u du=e^u+C$

7. $\int a^u du = \frac{a^u}{\ln a} + C$

8. $\int \sin u \cdot du = -\cos u + C$

9. $\int \cos u \cdot du = \sin u + C$

10. $\int \tan u \cdot du = -\ln|\cos u| + C$

11. $\int \cot u \cdot du = \ln|\sin u| + C$

12. $\int \sec u \cdot du = \ln|\sin u + \tan u| + C$

13. $\int \csc u \cdot du = -\ln|\cot u + \csc u| + C$

14. $\int \sec^2 u \cdot du = y \tan u + C$

15. $\int \csc^2 u \cdot du = -\cot u + C$

16. $\int \sec u \cdot \tan u \cdot du = \sec u + C$

17. $\int \csc u \cdot \cot u \cdot du = -\csc u + C$

18. $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\frac{a}{u} + C$

19. $\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$

20. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{|u|}{a} + C$

Ex.6 $\int \cot x \cdot \ln(\sin x) \cdot dx$

$\int u \cdot du = \frac{1}{2}u^2 + C = \frac{1}{2}[\ln(\sin x)]^2 + C$

令 $u = \ln \sin x$

$$du = \frac{\cos x}{\sin x} \cdot dx$$

$$du = \cot x \cdot dx$$

Ex: $\int \frac{x \cdot e^x}{(1+x)^2} \cdot dx$

Sol:令 $u = x \cdot e^x$

$$du = (x \cdot e^x + e^x)dx = e^x(1+x) \cdot dx$$

$$dv = \frac{dx}{(1+x)^2} = \frac{d(1+x)}{(1+x)^2}, v = \frac{1}{(1+X)}$$

$$\therefore 原式 = -\frac{x \cdot dx}{1+x} + \int \frac{e^x(1+x)}{(1+x)} \cdot dx$$

$$= -\frac{x \cdot e^x}{1+x} + e^x + C$$

$$= \frac{e^x}{1+x} + C$$

Chaper 8 infinite series

8.1 seqences

Definition of a Monotoic Sequence

A sequence $\{a_n\}$ is monotonic if its terms are mondecreasing

$$a_1 \leq a_2 \leq a_3 \leq .... \leq a_n \leq ......$$

or if its terms are nonincreasing

$$a_1 \geq a_2 \geq a_3 \geq a_4 \geq a_5 \geq .... \geq a_n \geq ....$$

Definition of Monotonic Sequence

Ex.8. Determine wheter each sequence having the given n^{th} term is monotonic

a. $a_n = 3 + (-1)^n$ b. $b_n = \frac{2n}{1+n}$

Sol: a. This sequence alternates between 2 and 4. Therefore , it is not monotonic

b. $b_n = \frac{2n}{1+n} \overset{?}{<} \frac{2(n+1)}{1+(n+1)} = b_{n+1}$

$2n(n+2) \overset{?}{<} 2(n+1)^2$

$2n^2 + 4n \overset{?}{<} 2^2 + 4n + 2$

$0 < 2(\text{O.K.})$

(each successive term is larger then its predecessor , this sequence is monotonic)

8.2 series and convergence

Definition of convergent and Divergent series

$S_n = a_1 + a_2 + a_3 + \ldots\ldots\ldots\ldots + a_n$ (n^{th} partial sum)

$\lim_{n\to\infty}$ = **有限值，則級數** $\sum_{n=1}^{\infty} a_n$ **收斂**

Then the series Σa_n converges

Ex.1. find the sun $\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots\ldots\ldots\ldots\ldots$

Sol: $S_1 = \frac{1}{2}$

$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$

$S_n = \ldots\ldots\ldots\ldots = \frac{2^n - 1}{2^n}$ $\therefore$ the series converges (Geometric Series)

8.3 The Integral Tsst and p-series

Theorem 8.11 convergence of p-series

The p-series

$$\sum_{n=1}^{\infty}\frac{1}{n^p}=\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\text{.........}$$

1. **convergence if p>1 , and**
2. **divergence if $0 < P \leq 1$**

Ex.3 Discuss The convergence or divergence of

(a). $\sum_{n=1}^{\infty}\frac{1}{n}$ (b). $\sum_{n=1}^{\infty}\frac{1}{n^2}$

Sol: (a) $\sum_{n=1}^{\infty}\frac{1}{n}$: p=1 $\therefore$ diveges

(b) $\sum_{n=1}^{\infty}\frac{1}{n^2}$: p=2 $\therefore$ convergence

8.4Compaarisons of series

Theorem 8.12 Direct comparison test

Let 0 $\leq a_n \leq b_n$ for all n

1. **if $\sum_{n=1}^{\infty} b_n$ Converges , then $\sum_{n=1}^{\infty} a_n$ Converges.**
2. **If $\sum_{n=1}^{\infty} a_n$ diverges , then $\sum_{n=1}^{\infty} b_n$ diverges.**

Ex.2 Determine the Convergence or divergence of $\sum_{n=1}^{\infty}\frac{1}{2+\sqrt{n}}$

Sol: 1. $2+\sqrt{n}>\sqrt{n}$

$$\therefore \frac{1}{2+\sqrt{n}}<\frac{1}{\sqrt{2}}$$

$\sum_{n=1}^{\infty}\frac{1}{\sqrt{2}}=\sum_{n=1}^{\infty}\frac{1}{n^{1/2}}$ diverges ,But $\frac{1}{2+\sqrt{n}}$ unknown

2. $n\geq 2+\sqrt{n}$ if $n\geq 4$

$\therefore \frac{1}{n}\leq\frac{1}{2+\sqrt{n}}$ if $n\geq 4$

$\sum_{n=1}^{\infty}\frac{1}{n}$ diverges $\therefore \sum_{n=1} 2+\sqrt{n}$ diverges

8.5 Alternating series 交錯級數

Theorem 8.14 Alternating series test

Let $a_n>0$, The alternating series

$\sum_{n=1}^{\infty}(-1)^n\cdot a_n$ **and** $\sum_{n=1}^{\infty}(-1)^{n+1}\cdot a_n$

Converge if the following two conditions are met

1. $\lim_{n\to\infty} a_n=0$ **2. $a_{n+1}<a_n$ for all n**

Ex.1 Determine the convergence or divergence of $\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{1}{n}$

Sol: $\lim_{n\to\infty}\frac{1}{n}=0 \quad a_{n+1}=\frac{1}{n+1}<\frac{1}{n}=a_n$

$\therefore$ the series converges

8.6 The Ratio Test

Theorem 8.17 Ratio Test

Let $\sum a_n$ **be series with nonzero terms**

1. $\sum a_n$ **converges absolutely if** $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|<1$

2. $\sum a_n$ **diverges if** $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|>1$ **or** $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\infty$

3. The Ratio test is inconclusive if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=1$

Ex.1 Determine the convergence or divergence of $\sum_{n=0}^{\infty}\frac{2n}{n!}$

Sol: $\lim_{n\to 0}\left|\frac{a_n+1}{a_n}\right|=\lim_{n\to 0}\left|\frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}}\right|=\lim_{n\to 0}\frac{2}{n+1}=0 \quad \therefore$ Converges